

Extremal Hilbert series and Gröbner bases

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Weak and Strong Lefschetz Properties across Mathematics
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- 2 The squarefree algebra case
- 3 Construction of the Gröbner basis

Standard graded algebras and monomial ideals

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We fix a monomial order $\prec$. Then,

- every non-zero polynomial of R has a leading monomial,
- and every non-zero ideal $I \subseteq R$ has
 - ▶ an initial monomial ideal $\text{in}_{\prec}(I)$
 - ▶ a unique reduced $\prec$ -Gröbner basis.

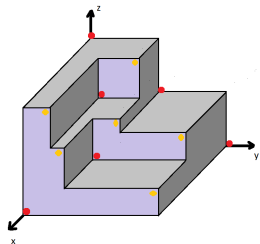


Figure: *Escalier* of a monomial ideal.

Hilbert series

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Let $M = \bigoplus_{d \geq 0} M_d$ be a finitely generated graded R -module with degree- d components M_d . The *Hilbert series* of M is

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For $A = \mathbf{k}[x_1, x_2]/(x_1^2, x_2^3)$, we have $\text{HS}(A; t) = 1 + 2t + 2t^2 + t^3$.

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Remark

Let $\{0\} \neq I \subseteq R$ be a homogeneous ideal, and $\prec$ a monomial ordering compatible with degrees. Then, $\text{HS}(R/I; t) = \text{HS}(R/\text{in}_{\prec}(I); t)$.

Open problems on Hilbert series

Given homogeneous ideal $I \subseteq R = \mathbf{k}[x_1, \dots, x_n]$ and homogeneous element $h \in R \setminus I$ of degree d , is it true that if the generators of I and h are generic of prescribed degrees, then all the multiplication maps $\cdot h : (R/I)_e \rightarrow (R/I)_{e+d}$ must have maximal rank?

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Fröberg's conjecture (1985)

The Hilbert series for the quotient of R by m general forms of degrees $a_1, \dots, a_m$ is

$$\left[\frac{\prod_{i=1}^m (1 - t^{a_i})}{(1 - t)^n} \right],$$

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Overarching goal

Understand graded ideals and algebras in generic position or defined by generic forms, using constructive methods of Computer Algebra.

A generic object

Definition

We study $I_{n,\mathbf{m},k} = (x_1^{m_1}, \dots, x_n^{m_n}, (x_1 + \dots + x_n)^k) \subset \mathbf{k}[x_1, \dots, x_n]$, where $\text{char}(\mathbf{k}) = 0$, $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 2}^n$ and $k \geq 1$.

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- These ideals and their algebras can be regarded as generic because they attain the Hilbert series of Fröberg's conjecture (Stanley 1980; Watanabe 1989).

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- These *almost complete intersections* have attracted much attention recently.
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- Our approach is the first to obtain explicit Gröbner bases and yields new, independent proof of Stanley's and Watanabe's results.

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Example and methods for squarefree algebra case

Write $I_{n,(2,\dots,2),k} = I_{n,k}$.

Example

The reduced Gröbner basis of $I_{5,2}$ is

$$G_{5,2} = \{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, \\ g_{\{1,2\},5,2}, g_{\{1,3,4\},5,2}, g_{\{2,3,4\},5,2}\},$$

where

$$g_{\{1,2\},5,2} = x_1x_2 + x_1x_3 + \cdots + x_4x_5,$$

$$g_{\{1,3,4\},5,2} = x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_3x_4x_5,$$

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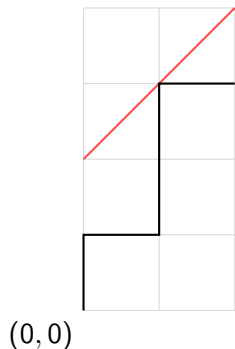
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Associate lattice paths to leading monomials.

$\text{in}(g_{\{1,3,4\},5,2}) = x_1x_3x_4$ is shown on the right.



Results for squarefree algebra

Theorem ([JLMOS 2024])

Consider the family $\mathcal{A}$ of subsets $A \subseteq [n]$ satisfying $\max(A) = 2|A| - k$ for some $k \geq 2$, and minimal with respect to inclusion.

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$$G_{n,k} = \{x_1^2, \dots, x_n^2\} \cup \bigcup_{d=k}^{k + \lfloor (n-k)/2 \rfloor} \{g_{A,n,k} \mid A \in \mathcal{A}, |A| = d\},$$

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where for $|A| = d$,

$$g_{A,n,k} = e_d(x_{i_1}, \dots, x_{i_{n-d+k}})$$

is the elementary symmetric polynomial of degree d in the variables indexed by the set $\{i_1, \dots, i_{n-d+k}\} = A \cup \{2d - k + 1, \dots, n\}$ with leading term $x_A = \prod_{a \in A} x_a$.

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Artinian monomial complete intersections and lattice paths

Monomial ideals $P_{n,\mathbf{m}} = (x_1^{m_1}, \dots, x_n^{m_n}) \subset R$ with $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 2}^n$ define Artinian complete intersection algebras $A_{n,\mathbf{m}} = R/P_{n,\mathbf{m}}$ which

- ① are Gorenstein,
- ② have the strong Lefschetz property,
- ③ and have palindromic, unimodal Hilbert series.

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Lemma ([JLMO 2026])

*Fix positive integers n , d and a vector $\mathbf{m} \in \mathbb{Z}_{\geq 2}^n$. Then the number of **$\mathbf{m}$ -admissible lattice paths ending at $(n, n - d)$ equals $\dim_{\mathbf{k}}(R/P_{n,\mathbf{m}})_d$.***

Number scheme and reflection points

For $1 \leq j \leq n$, let $\mathbf{m}_j = (m_1, \dots, m_j)$. We arrange the coefficients of $\text{HS}(\mathbf{k}; t)$, $\text{HS}(\mathbf{k}[x_1]/P_{1,\mathbf{m}_1}; t), \dots$ as columns in a number scheme.

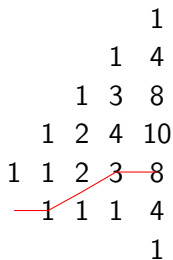


Figure: Here, $\mathbf{m} = (3, 2, 2, 3)$, and $k = 2$. A piecewise linear function interpolates between the points located $k/2$ below the midpoints of the columns.

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				1
		1		4
	1	3		8
1	2	4		10
1	1	2	3	8
1	1	1		4
				1

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Example

Let $\mathbf{m} = (3, 2, 2, 3)$ and $t = x_1 x_3 x_4^2$. Its path ends at the point $(4, 0) = (4, n - \deg(t))$, where it touches the red line.

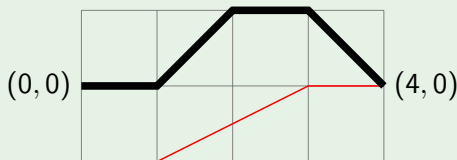


Figure: The graph of $L_{4,(3,2,2,3),2}$ and the lattice path associated with the monomial $x_1 x_3 x_4^2$.

Reflection and critical paths

Definition ([JLMO 2026])

Let $(a, b) \in \mathbb{Z}^2$ with $0 \leq a \leq n$. Reflection at the intersection of the red line and the vertical line $x = a$ yields $(a, b') =: (a, b)'$.

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$${}^i\mathbf{P}' := (P_0, P_1, \dots, P_{i-1}, P'_i, P'_{i+1}, \dots, P'_n)$$

is also **m**-admissible.

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$$\Lambda : \{\mathbf{P} \mid \mathbf{P} \text{ critical and } \mathbf{m}\text{-admissible}\} \rightarrow \{\mathbf{P} \mid \mathbf{P} \text{ } \mathbf{m}\text{-admissible}\},$$
$$\mathbf{P} \mapsto \lambda(\mathbf{P})\mathbf{P}'.$$

Critical path ideal

Definition ([JLMO 2026])

Let $M_{n,\mathbf{m},k}$ be the set of monomials corresponding to critical $\mathbf{m}$ -admissible lattice paths. We write $(M_{n,\mathbf{m},k})$ for the ideal generated by $M_{n,\mathbf{m},k} + P_{n,\mathbf{m}}$.

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Proposition ([JLMO 2026])

Let d be the degree of an $\mathbf{m}$ -free monomial and $d' = (\sum m_i) - n - d + k$. The degree-wise reflection map

$$\Lambda_d : \left\{ \begin{array}{l} \mathbf{P} \text{ critical with} \\ \text{endpoint } (n, n - d) \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \mathbf{P} \text{ critical with} \\ \text{endpoint } (n, n - d') \end{array} \right\}, \quad \mathbf{P} \mapsto \Lambda(\mathbf{P}).$$

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Minimal generators and Hilbert series

We partition the set of minimal critical paths,

$$\text{Crit}_{n,\mathbf{m},k} = \bigcup_{j=1}^n \text{Crit}_{n,\mathbf{m},k,j},$$

into subsets $\text{Crit}_{n,\mathbf{m},k,j}$ of generators having x_j as variable of highest index.

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Theorem ([JLMO 2026])

The ideal $(M_{n,\mathbf{m},k})$ is minimally generated by the set

$$S_{n,\mathbf{m},k} = \{x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n}\} \cup \text{Crit}_{n,\mathbf{m},k}.$$

where $x_j^{m_j}$ is removed if $k + \sum_{i=1}^{j-1} (m_i - 1) < m_j$.

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The Hilbert series of $R/(M_{n,\mathbf{m},k})$ and $R/P_{n,\mathbf{m}}$ satisfy

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where the brackets denote truncation at the first non-positive coefficient.

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Constructing g_s with $\text{in}(g_s) = s$ for every generator s of $(M_{n,\mathbf{m},k})$, obtain:

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Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2026])

Let p , q , and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r|pq$. Then

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The special role of the reverse lexicographic order

Remark ([JLMO 2026])

- If $n_1 < n_2$ and $\mathbf{m}_1 \in \mathbb{Z}_{\geq 2}^{n_1}$ is a prefix of $\mathbf{m}_2 \in \mathbb{Z}_{\geq 2}^{n_2}$, then $(M_{n_1, \mathbf{m}, k}) \subseteq (M_{n_2, \mathbf{m}, k})$.

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Thank you